J. Fluid Mech. (1975), *vol.* 71, *part* 4, *pp.* 711–727
Printed in Great Britain

Three-dimensional laminar compressible boundary layers with large injection

By C. S. VIMALA† AND G. NATH

Department of Applied Mathematics, Indian Institute of Science, Bangalore 560012

(Received 25 October 1974)

The effect of large mass injection on the following three-dimensional laminar compressible boundary-layer flows is investigated by employing the method of matched asymptotic expansions: (i) swirling flow in a laminar compressible boundary layer over an axisymmetric surface with variable cross-section and (ii) laminar compressible boundary-layer flow over a yawed infinite wing in a hypersonic flow. The resulting equations are solved numerically by combining the finite-difference technique with quasi-linearization. An increase in the swirl parameter, the yaw angle or the wall temperature is found to be capable of bringing the viscous layer nearer the surface and reducing the effects of massive blowing.

1. Introduction

An interesting aspect of the study of flows with mass addition is the phenomenon of boundary-layer blow-off. When large amounts of fluid are injected into a boundary layer, the injected fluid simply fills the region near the wall and causes significant alterations in the profiles of the flow variables. So, for large rates of injection, the boundary layer is characterized by (i) an inner layer close to the surface where the viscous forces are negligible compared with the pressure and inertia forces and (ii) a relatively thin outer viscous layer in which transition from the inner to the inviscid external flow takes place.

Owing to the presence of extremely small gradients and the largeness of the field of integration, the usual methods for handling two-point boundary conditions completely break down when the blowing parameter is very large. This failure can be attributed to the poor convergence and the instabilities of the numerical methods used. Since the prediction of the effects of massive blowing on slender aerodynamic bodies is of technological interest, it is desirable to carry out at least an approximate analysis of the structure of the boundary layer under conditions of strong blowing. In fact, several studies of boundary layers with large injection have been carried out.

Pretsch (1944), Watson (1966), Aroesty & Cole (1965) and Kubota & Fernandez (1968) have developed solutions to the Falkner–Skan equation or its compressible analogue for large rates of blowing. Kassoy (1971) has rederived

† Present address: Aerodynamics Division, National Aeronautical Laboratory, Bangalore.

the problem considered by Kubota & Fernandez (1968) directly from the original equations for the hard-blowing case. Nachtsheim & Green (1971) and Liu & Nachtsheim (1973) have proposed a purely numerical method through a conversion of the two-point boundary-value problem into a three-point boundary-value problem. The incompressible boundary-layer flow at a three-dimensional stagnation point with strong blowing has been investigated by Walton (1973) by the method of matched asymptotic expansions. The methods of Kassoy (1971), Nachtsheim & Green (1971) and Liu & Nachtsheim (1973) involve three-point boundary-value problems which can be solved only by an inverse method. On the other hand, in the method due to Kubota & Fernandez (1968), one faces only a two-point boundary-value problem, which can be solved directly.

So the method due to Kubota & Fernandez (1968) has been employed here to study the effect of large injection on (i) swirling flow in a laminar compressible boundary layer over an axisymmetric surface with variable cross-section and (ii) laminar compressible boundary-layer flow over a yawed infinite wing in a hypersonic flow. The method consists of constructing a uniformly valid approximate solution for the hard-blowing case by using the modified von Mises transformation and the method of matched asymptotic expansions. Since the usual shooting procedures become unstable for the strong-blowing case, the equations are solved numerically by combining the finite-difference technique with quasilinearization (Kubota & Fernandez 1968).

2. Swirling flow in a laminar compressible boundary layer over an axisymmetric surface with variable cross-section

2.1. *Analysis*

The similarity equations governing the low-speed swirling laminar compressible boundary-layer flow of a perfect gas with density ρ, constant specific heat c_p, viscosity μ proportional to the temperature T and Prandtl number unity caused by the superposition of a free vortex on the longitudinal flow over an axisymmetric surface of radius r with large injection at the surface are (Back 1969; Vimala 1974)

$$f''' + ff'' + \bar{\beta}[G(1-g_w) + g_w - f'^2] + \bar{\alpha}[G(1-g_w) + g_w - G^2] = 0, \qquad (2.1a)$$

$$G'' + fG' = 0, \qquad (2.1b)$$

with the boundary conditions

$$f(0) = f_w(\leqslant -1), \quad f'(0) = G(0) = 0, \quad f'(\infty) = G(\infty) = 1. \qquad (2.1c)$$

Here f is a dimensionless stream function defined in such a way that $f' = u/u_e$, where u and u_e are the longitudinal velocity components in the ξ direction inside and outside the boundary layer respectively. G stands for both the normalized swirl velocity component and the enthalpy difference ratio and is given by

$$G = v/v_e = (g-g_w)/(1-g_w), \qquad (2.2a)$$

where v is the swirl velocity in the η direction,

$$g = H/H_e, \quad H = c_pT + \tfrac{1}{2}(u^2+v^2) \qquad (2.2b)$$

and the suffixes w and e denote values at the wall and the edge of the boundary layer respectively. $\bar{\alpha}$, $\bar{\beta}$, g_w and f_w are the swirl, longitudinal acceleration, wall temperature and mass-transfer parameter respectively and are given by

$$\bar{\alpha} = \frac{2X}{r}\left(-\frac{dr}{dX}\right)\left(\frac{v_e}{u_e}\right)^2\left(\frac{H_e}{h_e}\right), \quad \bar{\beta} = \frac{2X}{u_e}\left(\frac{du_e}{dX}\right)\left(\frac{H_e}{h_e}\right), \tag{2.2c}$$

$$g_w = \frac{H_w}{H_e}, \quad f_w = \frac{-\rho_w w_w (2X)^{\frac{1}{2}}}{r\rho_e \mu_e u_e}, \tag{2.2d}$$

where

$$X = \int_0^\xi \rho_e \mu_e u_e r^2 d\xi, \quad h = c_p T \tag{2.2e}$$

and w is the velocity component normal to the surface, in the ζ direction. Primes denote differentiation with respect to the independent similarity variable Z defined as

$$Z = \frac{r\rho_e u_e}{(2X)^{\frac{1}{2}}}\int_0^\zeta \frac{\rho}{\rho_e} d\zeta. \tag{2.2f}$$

The solutions of (2.1) for the cases of zero and moderate mass transfer have already been obtained by Back (1969) and Vimala (1974) respectively. Defining a new dependent variable

$$W = f'^2 \tag{2.3}$$

and changing the independent variable from Z to f, (2.1) are transformed into

$$W^{\frac{1}{2}}\frac{d^2W}{df^2} + f\frac{dW}{df} + 2\bar{\beta}[G(1-g_w)+g_w - W] + 2\bar{\alpha}[G(1-g_w)+g_w - G^2] = 0, \tag{2.4a}$$

$$\frac{d}{df}\left(W^{\frac{1}{2}}\frac{dG}{df}\right) + f\frac{dG}{df} = 0, \tag{2.4b}$$

with the boundary conditions

$$W(f_w) = G(f_w) = 0, \quad W(\infty) = G(\infty) = 1. \tag{2.5}$$

In order to consider the large-injection case $-f_w \gg 1$, when the occurrence of blow-off phenomena is possible, it is convenient to divide the boundary layer into an inner and an outer region.

For the inner region, near the wall, introducing a new independent variable

$$\bar{f} = f/(-f_w) = \epsilon^{\frac{1}{2}} f \tag{2.6}$$

transforms (2.4) into

$$\epsilon W^{\frac{1}{2}}\frac{d^2W}{d\bar{f}^2} + \bar{f}\frac{dW}{d\bar{f}} + 2\bar{\beta}[G(1-g_w)+g_w - W] + 2\bar{\alpha}[G(1-g_w)+g_w - G^2] = 0, \tag{2.7a}$$

$$\epsilon\frac{d}{d\bar{f}}\left(W^{\frac{1}{2}}\frac{dG}{d\bar{f}}\right) + \bar{f}\frac{dG}{d\bar{f}} = 0, \tag{2.7b}$$

with the boundary conditions

$$W(-1) = G(-1) = 0, \quad W(\infty) = G(\infty) = 1. \tag{2.8}$$

For large negative values of f_w, i.e. for $-f_w \gg 1$,

$$\epsilon = (-f_w)^{-2} \ll 1. \tag{2.9}$$

So, on expanding W and G in terms of ϵ as

$$W = \overline{W}_0(\bar{f}) + \epsilon \overline{W}_1(\bar{f}) + \ldots, \tag{2.10a}$$

$$G = \bar{G}_0(\bar{f}) + \epsilon \bar{G}_1(\bar{f}) + \ldots \tag{2.10b}$$

and inserting these into (2.7), there results a power series in ϵ which must vanish identically for all ϵ. Assuming $\bar{\alpha}$ and $\bar{\beta}$ to be of order unity, we get the following sets of second-order differential equations at zeroth and first order in ϵ:

$$\left.\begin{aligned} &\bar{f}\,d\overline{W}_0/d\bar{f} + 2\bar{\beta}[\bar{G}_0(1-g_w) + g_w - \overline{W}_0] \\ &\qquad + 2\bar{\alpha}[\bar{G}_0(1-g_w) + g_w - \bar{G}_0^2] = 0 \quad (2.11a)\\ &\bar{f}\,d\bar{G}_0/d\bar{f} = 0 \quad (2.11b)\\ &\overline{W}_0(-1) = \bar{G}_0(-1) = 0 \quad (2.11c)\end{aligned}\right\} \text{ at zeroth order,}$$

$$\left.\begin{aligned} &\overline{W}_0^{\frac{1}{2}}\frac{d^2\overline{W}_0}{d\bar{f}^2} + \bar{f}\frac{d\overline{W}_1}{d\bar{f}} + 2\bar{\beta}[\bar{G}_1(1-g_w) - \overline{W}_1] \\ &\qquad + 2\bar{\alpha}[\bar{G}_1(1-g_w) - 2\bar{G}_0\bar{G}_1] = 0 \quad (2.12a)\\ &\frac{d}{d\bar{f}}\left[\overline{W}_0^{\frac{1}{2}}\frac{d\bar{G}_0}{d\bar{f}}\right] + \bar{f}\frac{d\bar{G}_1}{d\bar{f}} = 0 \quad (2.12b)\\ &\overline{W}_1(-1) = \bar{G}_1(-1) = 0 \quad (2.12c)\end{aligned}\right\} \text{ at } O(\epsilon).$$

Since the equations (2.11) and (2.12) resulting from an expansion near the wall as $\epsilon \to 0$ are of order two, while the original equations (2.7) are of order three, the boundary conditions as $\bar{f} \to \infty$ have been abandoned.

An examination of the energy equations and the boundary conditions for $\bar{G}_0$, $\bar{G}_1$, etc., gives immediately (Kubota & Fernandez 1968)

$$\bar{G}_0 = \bar{G}_1 = \ldots = 0. \tag{2.13}$$

Therefore the enthalpy difference ratio and the swirl velocity are zero in the inner layer. It may be noted that the same results are obtained by using the method analogous to that given by Walton (1973).†

The solutions for $\overline{W}_0$ and $\overline{W}_1$ are

$$\overline{W}_0 = (g_w/\bar{\beta})(\bar{\alpha}+\bar{\beta})[1-(-\bar{f})^{2\bar{\beta}}], \tag{2.14a}$$

$$\overline{W}_1 = [(g_w/\bar{\beta})(\bar{\alpha}+\bar{\beta})]^{\frac{1}{2}}\, 2\bar{\beta}(2\bar{\beta}-1)(-\bar{f})^{2\bar{\beta}} \int_1^{-\bar{f}} \frac{(1-t^{2\bar{\beta}})^{\frac{1}{2}}}{t^3}\,dt. \tag{2.14b}$$

Although the integral in the expression for $\overline{W}_1$ can be represented in terms of a hypergeometric function, in general little insight is gained by doing so. On the other hand, it is important to note that as $\bar{f} \to 0-$ the behaviour of $\overline{W}_1$ is given by (Watson 1966)

$$\overline{W}_1 \sim -[g_w(\bar{\alpha}+\bar{\beta})]^{\frac{1}{2}}(\bar{\beta})^{-\frac{1}{2}}(2\bar{\beta}-1)(-\bar{f})^{2\bar{\beta}-2}[1+O(E(-\bar{f}))], \tag{2.15a}$$

† The authors are grateful to a referee for suggesting this method.

where

$$E(-\bar{f}) = \begin{cases} (-\bar{f})^{2\bar{\beta}} & \text{if } \bar{\beta} < 1, \\ (-\bar{f})^2 \log(-\bar{f}) & \text{if } \bar{\beta} = 1, \\ (-\bar{f})^2 & \text{if } \bar{\beta} > 1. \end{cases} \tag{2.15b}$$

Equations (2.14) and (2.15) indicate that the inner solution cannot be continued past $\bar{f} = 0$ as it fails to satisfy the asymptotic boundary conditions. This discontinuity in W is smoothed by viscosity in a thin layer around $f = 0$. So, for the purpose of matching the inner solution to the outer uniform flow, a transitional expansion is necessary. In this transitional viscous region, where viscous terms and inviscid terms are of the same order, it is appropriate to use the physical variable f itself in the expansion. The behaviour of the inner solution as $\bar{f} \to 0-$ suggests an expansion of the form

$$W = W_0(f) + \epsilon^{\bar{\beta}} W_1(f) + \ldots, \tag{2.16a}$$

$$G = G_0(f) + \epsilon^{\bar{\beta}} G_1(f) + \ldots. \tag{2.16b}$$

Here it may be noted that the next term in the above expansion is proportional to $\epsilon^{2\bar{\beta}}$ only if $\bar{\beta} < 1$. For $\bar{\beta} = 1$ and $\bar{\beta} > 1$, it is proportional to $\epsilon^2 \log \epsilon$ and ϵ^2 respectively (Watson 1966). However, we have not included higher terms in the expansion beyond $\epsilon^{\bar{\beta}}$ as their contributions are considered to be small (Kubota & Fernandez 1968). It has been found that the results thus obtained (i.e. by neglecting higher terms beyond $\epsilon^{\bar{\beta}}$) are in good agreement with exact numerical results (see figure 1).

Substitution of (2.16) in (2.4) yields the following pairs of equations for

$$(W_0, G_0) \quad \text{and} \quad (W_1, G_1):$$

$$\left.\begin{aligned} &W_0^{\frac{1}{2}} \frac{d^2 W_0}{df^2} + f\frac{dW_0}{df} + 2\bar{\beta}[G_0(1-g_w) + g_w - W_0] \\ &\qquad + 2\bar{\alpha}[G_0(1-g_w) + g_w - G_0^2] = 0 \qquad (2.17a) \\ &\frac{d}{df}\left(W_0^{\frac{1}{2}} \frac{dG_0}{df}\right) + f\frac{dG_0}{df} = 0 \qquad (2.17b) \end{aligned}\right\} \text{at zeroth order,}$$

$$\left.\begin{aligned} &W_0^{\frac{1}{2}} \frac{d^2 W_1}{df^2} + \frac{W_1}{2W_0^{\frac{1}{2}}} \frac{d^2 W_0}{df^2} + f\frac{dW_1}{df} + 2\bar{\beta}[G_1(1-g_w) - W_1] \\ &\qquad + 2\bar{\alpha}[G_1(1-g_w) - 2G_0 G_1] = 0 \qquad (2.18a) \\ &\frac{d}{df}\left[W_0^{\frac{1}{2}} \frac{dG_1}{df} + \frac{W_1}{2W_0^{\frac{1}{2}}} \frac{dG_0}{df}\right] + f\frac{dG_1}{df} = 0 \qquad (2.18b) \end{aligned}\right\} \text{at } \quad O(\epsilon).$$

The boundary conditions for (2.17) and (2.18) are to be chosen such that the transition solution matches the outer (uniform flow) solution and the inner (inviscid) solution to some prescribed order in ϵ. In the outer region, where $f \to +\infty$, it is required that $W = G = 1$, which can be satisfied only if

$$W_0(\infty) = G_0(\infty) = 1, \quad W_1(\infty) = G_1(\infty) = 0. \tag{2.19}$$

Another set of boundary conditions may be obtained by using the requirement that an overlap domain exists where the solution to (2.17) and (2.18) matches the inner solution. To determine these boundary conditions, let an intermediate variable $f^* = \bar{f}/\nu(\epsilon)$ be introduced. Where $\nu(\epsilon) \to 0$ and $\nu(\epsilon)/\epsilon^{\frac{1}{2}} \to \infty$ as $\epsilon \to 0$ with f^* fixed, the limits of the inner and transition solutions match. This limit implies that $\bar{f} = \nu f^* \to 0-$ and $f = \nu f^*/\epsilon^{\frac{1}{2}} \to -\infty$ as $\epsilon \to 0$ for negative f^*.

With the help of (2.14) and (2.15), the inner solution near $\bar{f} = 0-$ can be written in terms of f^* as

$$W_{\text{in}}(f;\epsilon) \sim \frac{g_w}{\bar{\beta}}(\bar{\alpha}+\bar{\beta})[1-\nu^{2\bar{\beta}}(-f^*)^{2\bar{\beta}}]$$
$$-\frac{\epsilon}{\nu^2} f^{*-2\bar{\beta}-\frac{1}{2}}(2\bar{\beta}-1)[(\bar{\alpha}+\bar{\beta})g_w]^{\frac{3}{2}}(-\nu f^*)^{2\bar{\beta}}[1+O(E(-\nu f^*))], \quad (2.20a)$$

where $E(-\nu f^*)$ is defined by (2.15b), and

$$G_{\text{in}}(f;\epsilon) \sim 0. \quad (2.20b)$$

On the other hand, the transition solution can be expressed as

$$W_{\text{tr}} \sim W_0(\nu f^*/\epsilon^{\frac{1}{2}}) + \epsilon^{\bar{\beta}} W_1(\nu f^*/\epsilon^{\frac{1}{2}}) + \ldots, \quad (2.21a)$$

$$G_{\text{tr}} \sim G_0(\nu f^*/\epsilon^{\frac{1}{2}}) + \epsilon^{\bar{\beta}} G_1(\nu f^*/\epsilon^{\frac{1}{2}}) + \ldots, \quad (2.21b)$$

so that

$$W_{\text{tr}} - W_{\text{in}} = W_0(\nu f^*/\epsilon^{\frac{1}{2}}) - (g_w/\bar{\beta})(\bar{\alpha}+\bar{\beta}) + \nu^{2\bar{\beta}}[(\epsilon^{\frac{1}{2}}/\nu)^{2\bar{\beta}} W_1(\nu f^*/\epsilon^{\frac{1}{2}})$$
$$+ (g_w/\bar{\beta})(\bar{\alpha}+\bar{\beta})(-f^*)^{2\bar{\beta}}] + O(\epsilon/\nu^2), \quad (2.22a)$$

$$G_{\text{tr}} - G_{\text{in}} = G_0(\nu f^*/\epsilon^{\frac{1}{2}}) + \epsilon^{\bar{\beta}} G_1(\nu f^*/\epsilon^{\frac{1}{2}}) + \ldots. \quad (2.22b)$$

Hence, for matching to be successful, the conditions to be imposed on W_0, G_0, W_1 and G_1 as $f \to -\infty$ are

$$W_0(-\infty) = (g_w/\bar{\beta})(\bar{\alpha}+\bar{\beta}), \quad G_0(-\infty) = 0, \quad (2.23a)$$

$$W_1(-\infty) = (-g_w/\bar{\beta})(\bar{\alpha}+\bar{\beta})(-f)^{2\bar{\beta}}, \quad G_1(-\infty) = 0. \quad (2.23b)$$

Once the transition solution is available, the composite solution can be constructed by adding the two solutions and subtracting the common part, which happens to be the inner solution in this case. Therefore

$$W(f;f_w) = W_0(f) + (-f_w)^{-2\bar{\beta}} W_1(f), \quad (2.24a)$$

$$G(f;f_w) = G_0(f) + (-f_w)^{-2\bar{\beta}} G_1(f) \quad (2.24b)$$

is a uniformly valid solution. In (2.24), W_0, W_1, G_0 and G_1 may be obtained by solving (2.17) and (2.18) with the boundary conditions (2.19) and (2.23). It might also be noted that, in (2.24), W_0 represents the velocity in the viscous mixing layer while W_1 represents the rounding off of corners in the velocity profile. One more point worth noting is that W_0, W_1, G_0 and G_1 are independent of f_w, the blowing parameter, which enters only as a multiplicative factor in the final solution (2.24). Equations (2.17) and (2.18) do not possess any singularities and may be integrated numerically with the help of a stable numerical scheme which is essentially a combination of the quasi-linearization technique and the finite-difference technique (Kubota & Fernandez 1968; Vimala 1974).

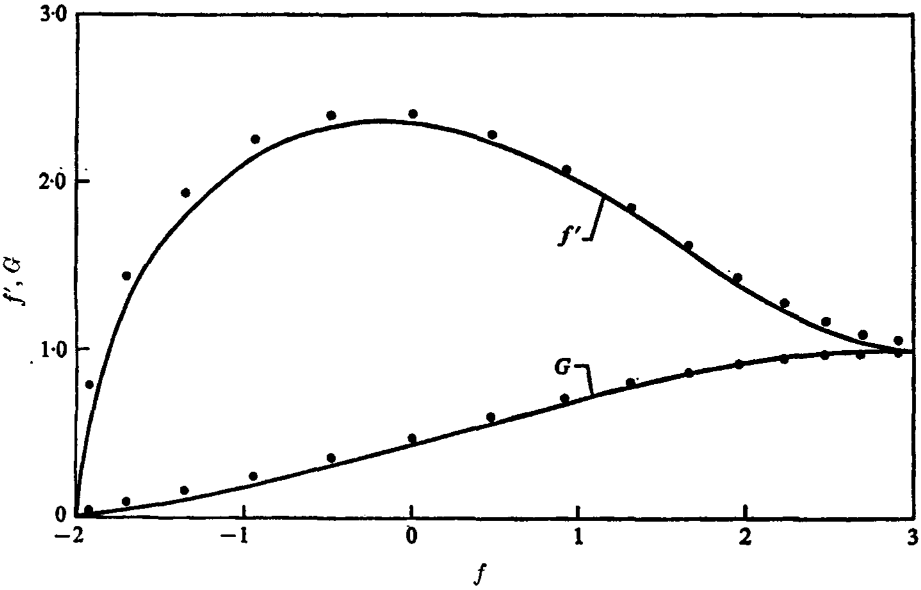

FIGURE 1. Comparison of the asymptotic solution with exact results (circles): velocity and enthalpy profiles (axisymmetric surface). $\bar{\alpha} = 10{\cdot}0$, $\bar{\beta} = 1{\cdot}0$, $f_w = -2{\cdot}0$, $g_w = 1{\cdot}0$.

The physical distance Z across the boundary layer is given by

$$Z = \int_{f_w}^{f} W^{-\frac{1}{2}}\, df = \int_{f_w}^{f_w+\Delta f} W^{-\frac{1}{2}}\, df + \int_{f_w+\Delta f}^{f} W^{-\frac{1}{2}}\, df. \tag{2.25}$$

The second integral can be evaluated numerically without any difficulty whereas a straightforward numerical integration of the first integral is very difficult, since $W \to 0$ as $f \to f_w$. However, the first integral can be calculated analytically with the help of the inner solution W_0, which is valid for small values of Δf. Using (2.14), we obtain

$$\int_{f_w}^{f_w+\Delta f} W^{-\frac{1}{2}}\, df = \left[\frac{g_w}{\bar{\beta}}(\bar{\alpha}+\bar{\beta})\right]^{-\frac{1}{2}} \int_{f_w}^{f_w+\Delta f} [1-(f/f_w)^{2\bar{\beta}}]^{-\frac{1}{2}}\, df$$

$$= \left[\frac{-2f_w\Delta f}{(\bar{\alpha}+\bar{\beta})\,g_w}\right]^{\frac{1}{2}} \quad \text{if} \quad \Delta f/-f_w \ll 1. \tag{2.26}$$

Hence

$$Z = \left[\frac{-2f_w\Delta f}{(\bar{\alpha}+\bar{\beta})\,g_w}\right]^{\frac{1}{2}} + \int_{f_w+\Delta f}^{f} W^{-\frac{1}{2}}\, df. \tag{2.27}$$

Next, the wall-shear-stress and the heat-transfer parameters f''_w and G'_w are given by

$$f''_w = \tfrac{1}{2}(dW/df)_w = g_w(\bar{\alpha}+\bar{\beta})/-f_w + O(\epsilon^2), \tag{2.28a}$$

$$G'_w = (f'dG/df)_w = 0 \quad \text{to all orders in } \epsilon \tag{2.28b}$$

on using (2.11) and (2.12). It can be concluded from (2.28) that, at least for the case of a Prandtl number of unity and $\mu \propto T$, for finite $\bar{\alpha}$ and $\bar{\beta}$ the heat transfer and the tangential shear stress at the wall vanish whereas the longitudinal

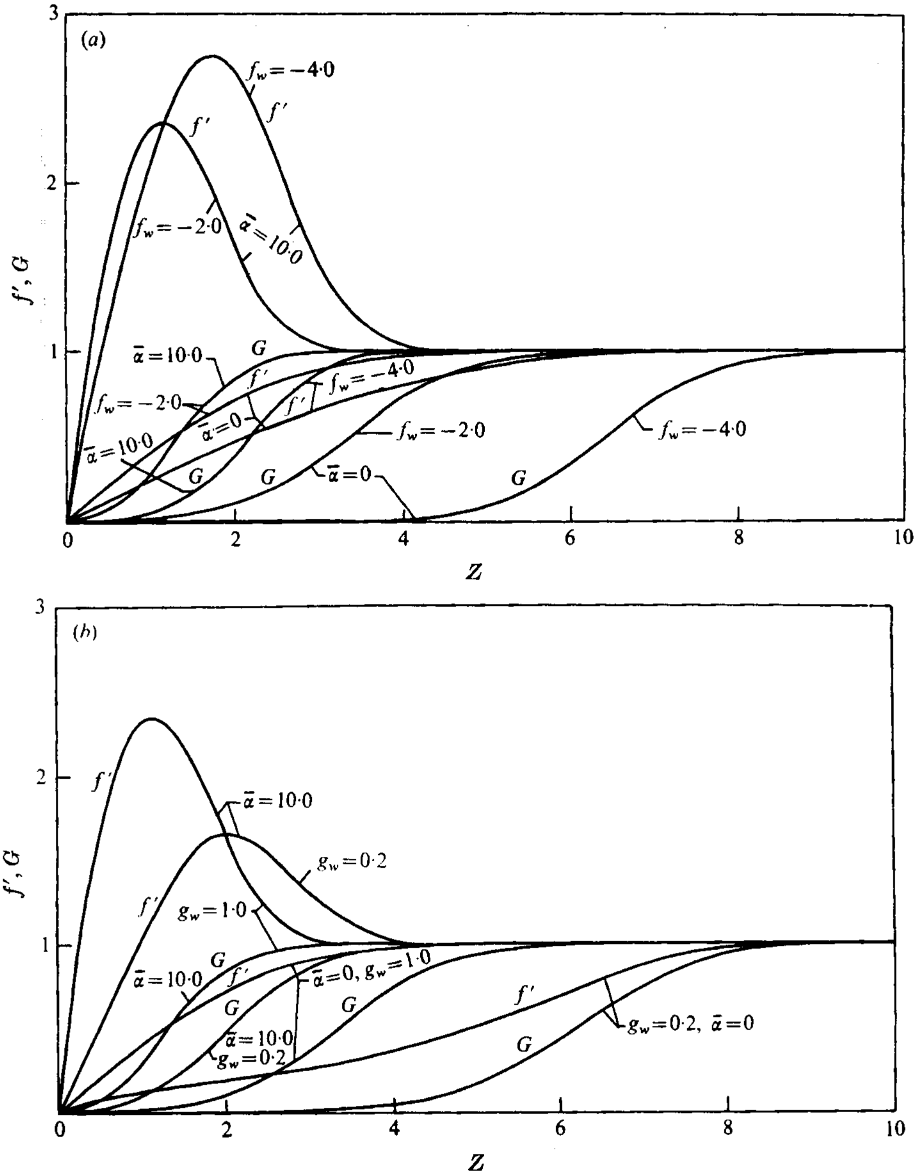

FIGURE 2. Velocity and enthalpy profiles for the case of large injection (axisymmetric surface). $\bar{\alpha} = 0$ or $10\cdot0$, $\bar{\beta} = 1\cdot0$. (*a*) $f_w = -2\cdot0$ or $-4\cdot0$, $g_w = 1\cdot0$. (*b*) $f_w = -2\cdot0$, $g_w = 0\cdot2$ or $1\cdot0$.

shear-stress component is still finite. Evidently, the mass leaving the wall convects away the heat generated at the wall by viscous dissipation.

2.2. *Results*

Figure 1 demonstrates the validity of this asymptotic analysis for large values of f_w by comparing the asymptotic solutions with the exact solutions obtained from a straightforward numerical integration of (2.1) with the boundary con-

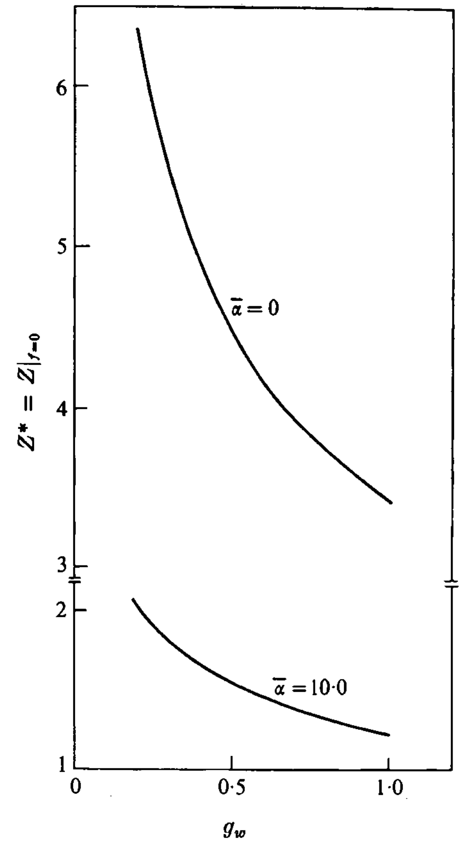

FIGURE 3. Effect of swirl and wall temperature on the location of the dividing streamline ($f = 0$) (axisymmetric surface). $\bar{\beta} = 1{\cdot}0, f_w = -2{\cdot}0$.

ditions (2.2) by the method of quasi-linearization (Vimala 1974) for the case $\bar{\alpha} = 10{\cdot}0$, $\bar{\beta} = 1{\cdot}0, f_w = -2{\cdot}0$. It can be seen that there is only a slight departure of the values obtained by this asymptotic analysis from the exact values.

In figures 2 (*a*) and (*b*), the effects of the wall temperature g_w and the injection parameter f_w on the velocity and enthalpy profiles corresponding to $\bar{\alpha} = 0$ or $10{\cdot}0$ and $\bar{\beta} = 1{\cdot}0$ are presented. The variation of the location of the dividing streamline ($f = 0$) given by $Z^* = Z|_{f=0}$ and the variation of the corresponding velocity and enthalpy values f'^* and G^* with g_w for $\bar{\alpha} = 0$ or 10 and $\bar{\beta} = 1$ are shown in figures 3 and 4 respectively. It is clear from these figures that the dividing streamline is further away from the wall for $\bar{\alpha} = 0$ than for $\bar{\alpha} = 10{\cdot}0$. It can also be observed that the shifting of the dividing streamline away from the boundary becomes less and less as g_w increases. Further, an increase in g_w and $\bar{\alpha}$ produces an increase in f'^* but a decrease in G^*, the effect on f'^* being more pronounced than that on G^*.

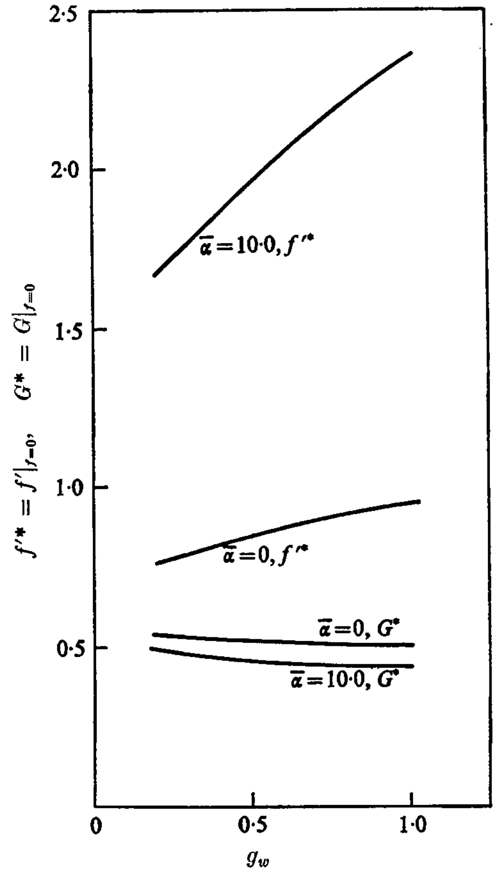

FIGURE 4. Effect of swirl and wall temperature on the dividing-streamline velocity and enthalpy (axisymmetric surface). $\bar{\beta} = 1\cdot0, f_w = -2\cdot0$.

3. Laminar compressible boundary-layer flow over a yawed infinite wing in a hypersonic flow

3.1. *Analysis*

The similarity equations governing the development of a laminar compressible boundary layer over a yawed infinite wing in a hypersonic flow of a perfect gas with density ρ, Prandtl number unity, viscosity μ proportional to the temperature T, kinematic viscosity ν and specific heat c_p with large injection at the surface are the following (Reshotko & Beckwith 1957; Whalen 1959; Vimala 1974):

$$f''' + ff'' = \beta[f'^2 - 1 + \tan^2\Lambda(G^2 - 1) + \sec^2\Lambda(\alpha - 1)(G - 1)], \quad (3.1a)$$

$$G'' + fG' = 0, \quad (3.1b)$$

with boundary conditions

$$f(0) = f_w \quad (\ll -1), \quad f'(0) = G(0) = 0, \quad f'(\infty) = G(\infty) = 1. \quad (3.1c)$$

In the above equations, f is a dimensionless stream function defined such that $f' = u/u_e$, where u and u_e are chordwise velocity components (in the x direction) inside and outside the boundary layer respectively. G represents both the normalized spanwise velocity component and the enthalpy difference ratio and is given by

$$G = v/v_e = (H-H_w)/(H_e-H_w), \tag{3.2a}$$

where

$$H = c_p T + \tfrac{1}{2}(u^2+v^2), \tag{3.2b}$$

v is the spanwise velocity component (in the y direction) and the suffixes w and e refer to values at the wall and the edge of the boundary layer respectively. Λ is the yaw angle; α, β and f_w are respectively the wall temperature, pressure-gradient and mass-transfer parameters and are given by

$$\alpha = T_w/T_0, \quad \beta = (\gamma-1)/\gamma, \tag{3.2c}$$

$$f_w = -w_w\left[\frac{\nu_w}{\beta}\left(1+\frac{\gamma-1}{2}\frac{u_e^2}{a_e^2}\frac{du_e}{dx}\right)\right]^{-\frac{1}{2}}, \tag{3.2d}$$

where w is the normal velocity component (along the z axis), a the sonic velocity, γ the ratio of specific heats and the subscript zero refers to a free-stream stagnation value. Further, primes denote differentiation with respect to the independent similarity variable η given by

$$\eta = Z\left(\frac{m+1}{2}\frac{\rho_e u_e}{\rho_0 \nu_0 X}\right)^{\frac{1}{2}}, \tag{3.2e}$$

where

$$m = \frac{\gamma-1}{\gamma+1}, \quad X = \int_0^x \frac{\mu_w}{\mu_0}\frac{T_0}{T_w}\frac{a_e}{a_0}\frac{p_e}{p_0}\,dx, \quad Z = \frac{a_e}{a_0}\int_0^z \frac{\rho}{\rho_0}\,dz. \tag{3.2f}$$

The nonlinear two-point boundary-value problems posed by 3.1) for the cases of zero and moderate mass transfer have been solved by Whalen (1959) and Vimala (1974) respectively. Here, for the purpose of considering the large-injection case, letting $W = f'^2$ as in §2 transforms (3.1) into

$$W^{\frac{1}{2}}\frac{d^2W}{df^2} + f\frac{dW}{df} = 2\beta[W-1+\tan^2\Lambda(G^2-1)+\sec^2\Lambda(\alpha-1)(G-1)], \tag{3.3a}$$

$$\frac{d}{df}\left(W^{\frac{1}{2}}\frac{dG}{df}\right) + f\frac{dG}{df} = 0, \tag{3.3b}$$

$$W(f_w) = G(f_w) = 0, \quad W(\infty) = G(\infty) = 1. \tag{3.3c}$$

For large injection rates, i.e. $f_w \ll -1$, it will be convenient to divide the boundary layer into an inner and an outer region. The wall-layer solutions are governed by the equations

$$\epsilon W^{\frac{1}{2}}\frac{d^2W}{d\bar{f}^2} + \bar{f}\frac{dW}{d\bar{f}} = 2\beta[(W-1)+\tan^2\Lambda\,(G^2-1)+\sec^2\Lambda(\alpha-1)(G-1)], \tag{3.4a}$$

$$\epsilon\frac{d}{d\bar{f}}\left(W^{\frac{1}{2}}\frac{dG}{d\bar{f}}\right) + \bar{f}\frac{dG}{d\bar{f}} = 0, \tag{3.4b}$$

with the boundary conditions

$$W(-1) = G(-1) = 0, \quad W(\infty) = G(\infty) = 1, \tag{3.5}$$

where

$$\bar{f} = f/-f_w = \epsilon^{\frac{1}{2}}f. \tag{3.6}$$

Since $-f_w \gg 1$, so that $\epsilon = (-f_w)^{-2} \ll 1$, W and G can be expanded in terms of ϵ as

$$W = \overline{W}_0(\bar{f}) + \epsilon \overline{W}_1(\bar{f}) + \ldots, \tag{3.7a}$$

$$G = \bar{G}_0(\bar{f}) + \epsilon \bar{G}_1(\bar{f}) + \ldots, \tag{3.7b}$$

where $(\overline{W}_0, \bar{G}_0)$ and $(\overline{W}_1, \bar{G}_1)$ satisfy the following zeroth-order and first-order equations:

$$\left.\begin{aligned} \bar{f}\, d\overline{W}_0/d\bar{f} &= 2\beta[\overline{W}_0 - 1 + \tan^2\Lambda(G_0^2 - 1) \\ &\quad + \sec^2\Lambda(\alpha - 1)(\bar{G}_0 - 1)] && (3.8a) \\ \bar{f}\, d\bar{G}_0/d\bar{f} &= 0 && (3.8b) \\ \overline{W}_0(-1) &= \bar{G}_0(-1) = 0 && (3.8c) \end{aligned}\right\} \text{ at zeroth order;}$$

$$\left.\begin{aligned} \overline{W}_0^{\frac{1}{2}} \frac{d^2\overline{W}_0}{d\bar{f}^2} + \bar{f}\frac{d\overline{W}_1}{d\bar{f}} &= 2\beta[\overline{W}_1 + 2\tan^2\Lambda\, \bar{G}_0\bar{G}_1 + \sec^2\Lambda(\alpha - 1)(\bar{G}_1)] && (3.9a) \\ \frac{d}{d\bar{f}}\left[\overline{W}_0^{\frac{1}{2}} \frac{d\bar{G}_0}{d\bar{f}}\right] + \bar{f}\frac{d\bar{G}_1}{d\bar{f}} &= 0 && (3.9b) \\ \overline{W}_1(-1) = \bar{G}_1(-1) &= 0 && (3.9c) \end{aligned}\right\} \text{ at } O(\epsilon).$$

The solutions to these sets of equations are (see Kubota & Fernandez 1968)

$$\bar{G}_0 = \bar{G}_1 = 0, \tag{3.10a}$$

$$\overline{W}_0 = \alpha \sec^2\Lambda[1 - (-\bar{f})^{2\beta}], \tag{3.10b}$$

$$\overline{W}_1 = (\alpha\sec^2\Lambda)^{\frac{1}{2}}\, 2\beta(2\beta - 1)(-\bar{f})^{2\beta} \int_1^{-\bar{f}} \frac{(1 - t^{2\beta})^{\frac{1}{2}}}{t^3}\, dt$$

$$\sim -(\alpha\sec^2\Lambda)^{\frac{1}{2}}\, \beta(2\beta - 1)(-\bar{f})^{2\beta - 2}[1 + O(E(-\bar{f}))] \quad (\text{for } \bar{f} \to 0-), \tag{3.10c}$$

where $E(-\bar{f})$ is given by (2.15*b*). It should be mentioned that, as in the problem in §2, the method analogous to that given by Walton (1973) also gives the result (3.10*a*).

The wall-shear-stress and heat-transfer parameters f_w'' and G_w'' are determined by the above inner-layer solutions and are given by

$$f_w'' = \tfrac{1}{2}(dW/df)_w = \alpha\beta\sec^2\Lambda/-f_w + O(\epsilon^2), \tag{3.11a}$$

$$G_w' = (f'\, dG/df)_w = 0. \tag{3.11b}$$

It is obvious from (3.11) that the wall-shear-stress component in the chordwise direction remains finite while the spanwise wall shear stress and the heat transfer at the surface are zero for large injection rates.

By developing an outer viscous-layer solution for the purpose of matching the inner solutions with the external inviscid solution through a procedure similar to that employed in §2, the uniformly valid solution

$$W = W_0(f) + (-f_w)^{-2\beta}\, W_1(f), \quad G = G_0(f) + (-f_w)^{-2\beta}\, G_1(f) \tag{3.12}$$

can be obtained for large injection rates. W_0, W_1, G_0 and G_1 are the solutions of the following sets of differential equations, which are numerically integrable

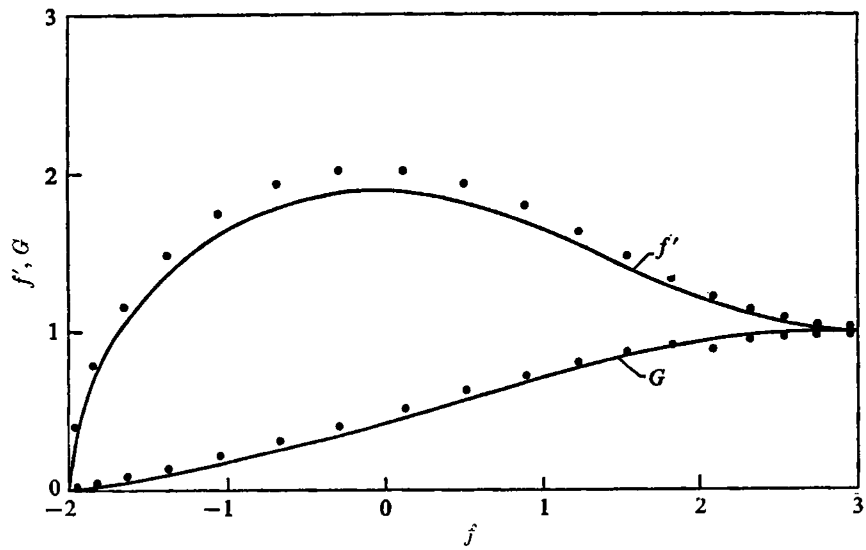

FIGURE 5. Comparison of the asymptotic solution with exact results (circles): velocity and enthalpy profiles (yawed infinite wing). $\alpha = 1{\cdot}0$, $\beta = 0{\cdot}2857$, $\Lambda = 75°$, $f_w = -2{\cdot}0$.

by combining quasi-linearization and finite-difference techniques (Kubota & Fernandez 1968; Vimala 1974):

$$\left.\begin{aligned} W_0^{\frac{1}{2}}\frac{d^2W_0}{df^2}+f\frac{dW_0}{df} &= 2\beta[(W_0-1)+\tan^2\Lambda(G_0^2-1) \\ &\qquad +\sec^2\Lambda(\alpha-1)(G_0-1)] \qquad (3.13a)\\ \frac{d}{df}\left(W_0^{\frac{1}{2}}\frac{dG_0}{df}\right)+f\frac{dG_0}{df} &= 0 \qquad (3.13b)\\ W_0(\infty)=G_0(\infty)=1,\quad W_0(-\infty)&=\alpha\sec^2\Lambda,\quad G_0(-\infty)=0 \qquad (3.13c)\end{aligned}\right\}\text{ at zero order;}$$

$$\left.\begin{aligned} W_0^{\frac{1}{2}}\frac{d^2W_1}{df^2}+\frac{W_1}{2W_0^{\frac{1}{2}}}\frac{d^2W_0}{df^2}+f\frac{dW_1}{df} & \\ = 2\beta[W_1+2\tan^2\Lambda\, G_0G_1+\sec^2\Lambda\,(\alpha-1)\,G_1] & \qquad (3.14a)\\ \frac{d}{df}\left[W_0^{\frac{1}{2}}\frac{dG_1}{df}+\frac{W_1}{2W_0^{\frac{1}{2}}}\frac{dG_0}{df}\right]+f\frac{dG_1}{df}=0 & \qquad (3.14b)\\ W_1(\infty)=G_1(\infty)=G_1(-\infty)=0, & \\ W_1(-\infty)=-\alpha\sec^2\Lambda(-f)^{2\beta} & \qquad (3.14c)\end{aligned}\right\}\text{ at } O((-f_w)^{-2\beta}).$$

Inversion of the transformation $W = f'^2$ yields the independent similarity variable η as

$$\eta = \int_{f_w}^{f} W^{-\frac{1}{2}}df = \left[\frac{-2f_w\Delta f}{\alpha\beta}\right]^{\frac{1}{2}}\cos\Lambda + \int_{f_w+\Delta f}^{f} W^{-\frac{1}{2}}df, \qquad (3.15)$$

where $\Delta f \ll 1$.

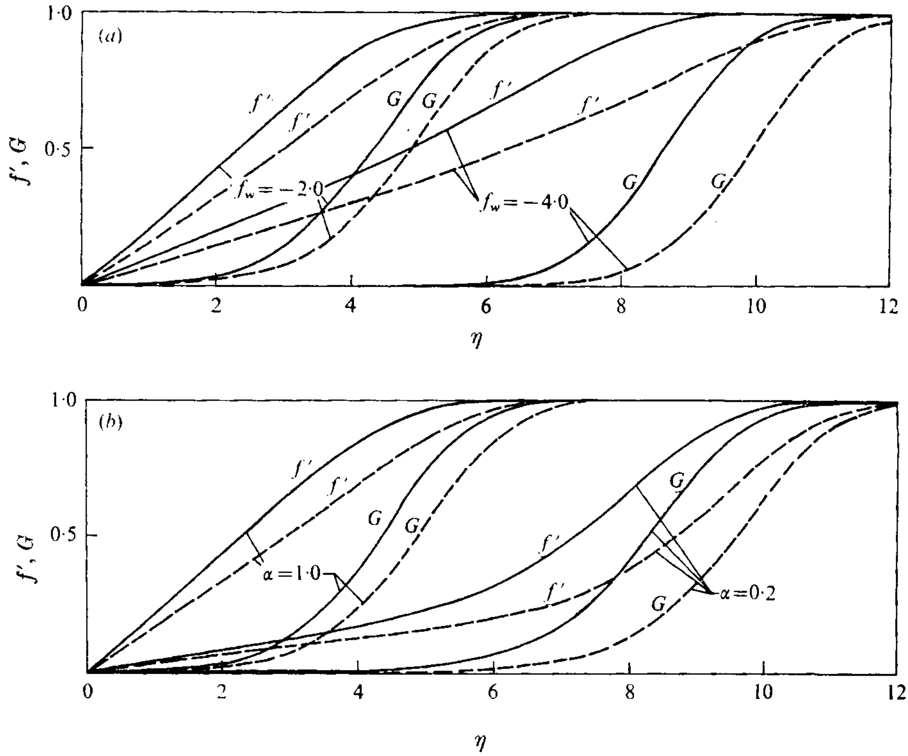

FIGURE 6. Velocity and enthalpy profiles for the case of large injection (yawed infinite wing). ——, $\Lambda = 30°$; – – , $\Lambda = 0$, $\beta = 0·2857$. (a) $\alpha = 1·0$, $f_w = -2·0$ or $-4·0$. (b) $\alpha = 0·2$ or $1·0, f_w = -2·0$.

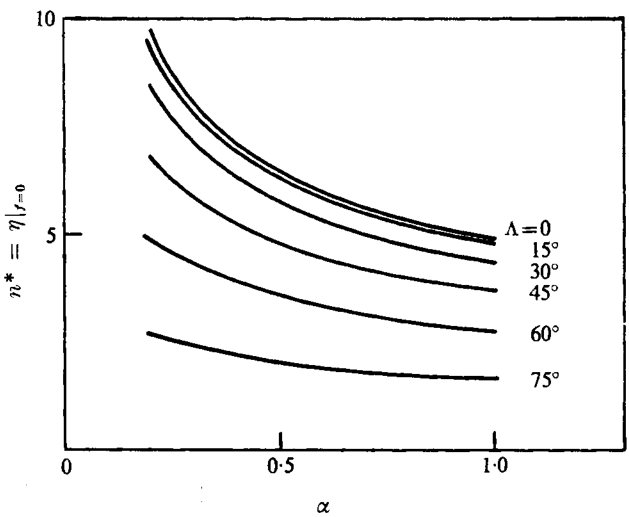

FIGURE 7. Effect of yaw and wall temperature on the location of the dividing streamline ($f = 0$) (yawed infinite wing). $\beta = 0·2857, f_w = -2·0$.

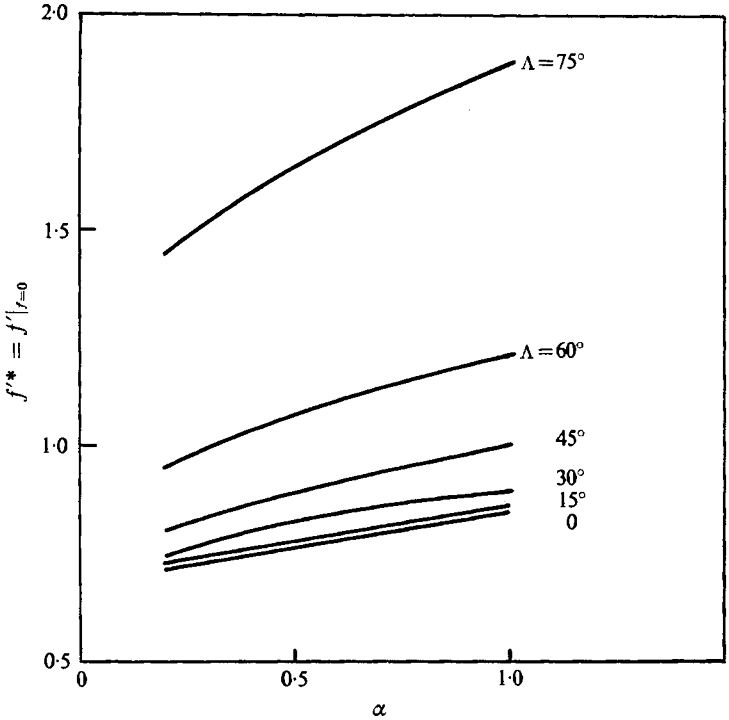

FIGURE 8. Effect of yaw and wall temperature on the dividing-streamline velocity in the chordwise direction (yawed infinite wing). $\beta = 0{\cdot}2857, f_w = -2{\cdot}0$.

3.2. *Results*

The asymptotic solutions resulting from (3.12) for the case $\beta = 0{\cdot}2857$, $\alpha = 1{\cdot}0$, $\Lambda = 75°, f_w = -2{\cdot}0$ have been compared with those obtained by quasi-linearization in figure 5. The departure of the approximate asymptotic solutions from the exact results is not appreciable, which proves that the present analysis holds good when f_w assumes larger and larger negative values. The influence of large injection on the velocity and enthalpy profiles is shown in figures 6(*a*) and (*b*). The variation of the location of the dividing streamline $f = 0$ and the corresponding velocity and enthalpy values at $f = 0$, namely η^*, f'^* and G^*, with α and Λ at large blowing rates is presented in figures 7–9.

It is found that η^* and G^* decrease as α and Λ increase whereas f'^* increases with Λ and α. So the boundary-layer blow-off phenomenon is observed more for small values of Λ and α. An increase in Λ or α or both is capable of bringing the dividing streamline nearer the boundary, at the same time increasing the chordwise velocity but decreasing the spanwise velocity and enthalpy at $f = 0$. However, the influence of α and Λ on G^* is less pronounced than that on η^* or f'^*.

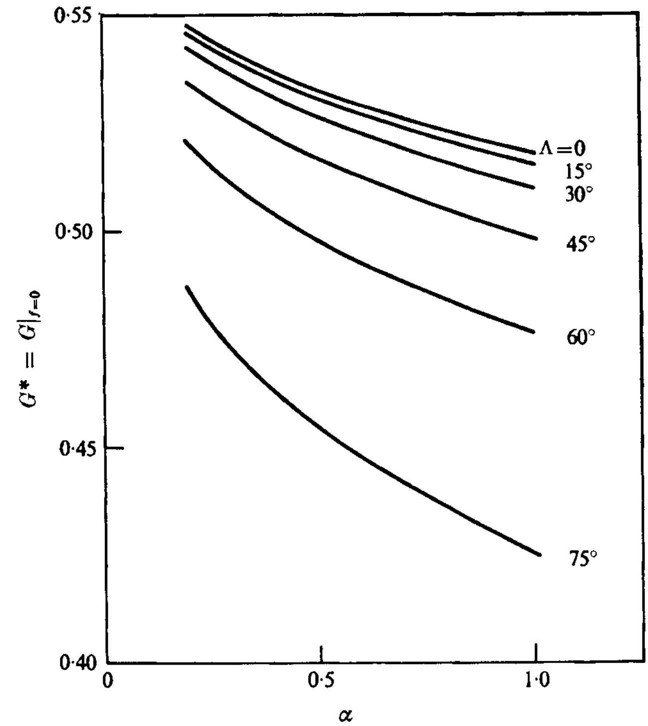

FIGURE 9. Effect of yaw and wall temperature on the dividing-streamline enthalpy and the dividing-streamline velocity in the spanwise direction (yawed infinite wing). $\beta = 0{\cdot}2857$, $f_w = -2{\cdot}0$.

4. Concluding remarks

It has been found that large rates of injection produce an increase in the velocity overshoot and blow the viscous layer away from the wall. In flows with swirl, the longitudinal wall-shear-stress parameter remains finite and is directly proportional to the sum of the swirl and the longitudinal pressure-gradient parameter and also to the wall temperature but inversely proportional to the injection parameter. On the other hand, both the tangential shear-stress parameter and the heat-transfer parameter vanish at the wall owing to viscous dissipation. An increase in the swirl parameter or the wall temperature or both is capable of bringing the viscous layer nearer the wall and at the same time increasing the longitudinal velocity component and decreasing the swirl velocity as well as the total enthalpy at the dividing streamline.

In the case of flow over yawed wings, similar behaviour is observed at large blowing rates. Even though the spanwise wall-shear-stress component and the heat transfer at the surface reduce to zero, the chordwise wall-shear-stress component is finite. In fact it is directly proportional to the wall temperature, the pressure gradient and the square of the secant of the yaw angle but inversely proportional to the injection parameter. Furthermore, it was found that the viscous layer can be shifted towards the wall and that the effect of an incipient blow-off phenomenon is reduced by increasing either the yaw angle or the wall temperature or both.

One of the authors (C.S.V.) wishes to thank the Council of Scientific and Industrial Research, New Delhi, and the Indian Institute of Science, Bangalore, for financial assistance.

REFERENCES

AROESTY, J. & COLE, J. D. 1965 Boundary-layer flows with large injection rates. *Rand Corp. Rep.* RM 4620.

BACK, L. H. 1969 Flow and heat transfer in laminar boundary layers with swirl. *A.I.A.A. J.* **7**, 1781.

KASSOY, D. R. 1971 On laminar boundary-layer blow-off. *J. Fluid Mech.* **48**, 209.

KUBOTA, T. & FERNANDEZ, F. L. 1968 Boundary layer flows with large injection and heat transfer. *A.I.A.A. J.* **6**, 22.

LIU, T. M. & NACHTSHEIM, P. R. 1973 Shooting method for solution of boundary layer flows with massive blowing. *A.I.A.A. J.* **11**, 1584.

NACHTSHEIM, P. R. & GREEN, M. J. 1971 Numerical solution of boundary-layer flows with massive blowing. *A.I.A.A. J.* **9**, 533.

PRETSCH, J. 1944 Analytic solution of the laminar boundary layer with asymptotic suction and injection. *Z. angew. Math. Mech.* **24**, 264.

RESHOTKO, E. & BECKWITH, I. E. 1957 Compressible laminar boundary layer over a yawed infinite cylinder with heat transfer and arbitrary Prandtl number. *N.A.C.A. Tech. Note*, no. 3986.

VIMALA, C. S. 1974 Flow problems in laminar compressible boundary layers. Ph.D. thesis, Indian Institute of Science, Bangalore.

WALTON, I. C. 1973 Boundary-layer flows at a three-dimensional stagnation point with strong blowing. *Quart. J. Mech. Appl. Math.* **16**, 413.

WATSON, E. J. 1966 The equation of similar profiles in boundary-layer theory with strong blowing. *Proc. Roy. Soc.* A **294**, 208.

WHALEN, R. J. 1959 Boundary-layer interaction on a yawed infinite wing in hypersonic flow. *J. Aerospace Sci.* **26**, 839.